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A harmonic oscillator on the Poincaré disc and hypercontractivity

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Abstract. Based on the geometry of a Poincaré disc, we construct a relativistic analogue of quantum mechanical harmonic oscillator with a hyperbolic phase space. The Hamiltonian is closely related to the ultraspherical operator and enjoys hypercontractivity. In the large-radius and large-spin limit, we recover the ordinary harmonic oscillator.

1. A harmonic oscillator in Euclidean space

The quantum mechanical harmonic oscillator is essentially the Weyl representation of the Euclidean motion group (or rather, its Lie algebra). In Fock–Bargmann model, it can be described by the quadruple [1]

$$\{H^2(C),\ \partial,\ \partial^*,\ H\}$$

where

$$H^2(C) = \left\{ f : C \to C, \text{holomorphic}, \|f\|^2 := \int_C f(z)\overline{f(z)}\pi^{-1}\mathrm{e}^{-z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z} < \infty \right\}$$

$$\partial f(z) = \frac{\partial}{\partial z} f(z) \qquad \partial^* f(z) = zf(z) \qquad Hf(z) = z\frac{\partial}{\partial z} f(z).$$

They satisfy the canonical commutation relations (CCR)

$$[\partial,\ \partial^*] = I \qquad [\partial,\ I] = 0 \qquad [\partial^*,\ I] = 0$$

and Wigner commutation relations

$$[\partial,\ H] = \partial \qquad [\partial^*,\ H] = -\partial^*.$$

The Hamiltonian $H = \partial^*\partial = z\partial/\partial z$ is diagonalized by the orthonormal basis $\{z^n/\sqrt{n!} : n \geqslant 0\}$ and has spectrum $\{0, 1, 2, \ldots\}$. It is remarkable that the semi-group $\{\mathrm{e}^{-tH} : t \geqslant 0\}$ enjoys hypercontractivity property [11], i.e. for appropriate t (precisely, $\mathrm{e}^{-2t} \leqslant p/q$), e^{-tH} is a contraction from $H^p(C)$ to $H^q(C)$, where for $p \geqslant 1$, $H^p(C)$ is the Banach space of holomorphic L^p functions on $\{C,\ \mathrm{d}\nu(z) = \pi^{-1}\mathrm{e}^{-z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}\}$.

Hypercontractivity plays an imortant role in the study of the Bose field [9, 10], see [3, 4] for surveys of this feature.

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2. A harmonic oscillator on the Poincaré disc

We shall introduce a simple quantization of the Poincaré disc from the discrete series of SU(1, 1) [2, 7]. Since SU(1, 1) is locally isomorphic to SO(2, 1), the Lorentz group which appears in the motion group in $1+2$ spacetime of special relativity, the model thus constructed will have a relativistic nature.

Let $D_r = \{z \in C : |z| < r\}$ be the Poincaré disc with radius r; it can be viewed as one sheet of the hyperbolic surface in R^3 and a Kähler manifold of hyperbolic nature. The geometric symmetry group of D_r is

$$G_r = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C) : g^* \begin{pmatrix} 1/r & 0 \\ 0 & -r \end{pmatrix} g = \begin{pmatrix} 1/r & 0 \\ 0 & -r \end{pmatrix} \right\}$$

$$= A\,\mathrm{SU}(1,1) A^{-1}$$

where

$$\mathrm{SU}(1,1) = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 = 1 \right\} \qquad A = \begin{pmatrix} \sqrt{r} & 0 \\ 0 & 1/\sqrt{r} \end{pmatrix}.$$

The G_r act on D_r via the fractional linear transfomation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} 1/r & 0 \\ 0 & -r \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1/r & 0 \\ 0 & -r \end{pmatrix}$$

is equivalent to

$$\begin{pmatrix} a\overline{a}/r - rc\overline{c} & \overline{a}b/r - r\overline{c}d \\ a\overline{b}/r - rc\overline{d} & b\overline{b}/r - rd\overline{d} \end{pmatrix} = \begin{pmatrix} 1/r & 0 \\ 0 & -r \end{pmatrix}.$$

When $r \to \infty$, we have $c = 0$, $a\overline{a} = 1$, $d\overline{d} = 1$, but

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} z = ad^{-1}z + bd^{-1}$$

hence G_r contracts to $E(C)$ (the Euclidean group on C) when $r \to \infty$.

The G_r-invariant measure on D_r is $\mathrm{d}\mu_r(z,\overline{z}) = \mathrm{d}z\mathrm{d}\overline{z}/(1 - z\overline{z}/r^2)^2$. For any $\lambda > 1$

$$\mathrm{d}\mu_{r,\lambda}(z,\overline{z}) = \frac{\lambda - 1}{\pi r^2}\left(1 - \frac{z\overline{z}}{r^2}\right)^{\lambda} \mathrm{d}\mu_r(z,\overline{z})$$

$$= \frac{\lambda - 1}{\pi r^2}\left(1 - \frac{z\overline{z}}{r^2}\right)^{\lambda-2} \mathrm{d}z\mathrm{d}\overline{z}$$

is a probability measure on D_r. Since $\mathrm{d}\mu_{r,\lambda} \longrightarrow \mathrm{d}\nu = \pi^{-1}\mathrm{e}^{-z\overline{z}}\mathrm{d}z\mathrm{d}\overline{z}$ when $\lambda/r^2 \to 1$ and $r \to \infty$, $\{D_r, \mu_{r,\lambda}\}$ may be viewed as a hyperbolic deformation of the one dimensional Gaussian space $\{C, \nu\}$ which is the phase space of harmonic oscillator of one degree freedom.

For $p \geqslant 1$, let

$$H_\lambda^p(D_r) = \left\{ f : D_r \to C, \text{holomorphic}, \|f\|_p^p = \int_{D_r} |f(z)|^p \,\mathrm{d}\mu_{r,\lambda}(z,\overline{z}) < \infty \right\}$$

then $H^p_\lambda(D_r)$ is a Banach space and $H^2_\lambda(D_r)$ is a reproducing kernel Hilbert space with kernel $k_\lambda(z,w) = (1 - z\overline{w}/r^2)^{-\lambda}$, i.e.

$$f(z) = \int_{D_r} f(w) k_\lambda(z,w)\, \mathrm{d}\mu_{r,\lambda}(w,\overline{w}) \qquad \forall\ f \in H^2_\lambda(D_r).$$

An orthonormal basis of $H^2_\lambda(D_r)$ is

$$\left\{ e_n(z) = \sqrt{\frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)}}\frac{z^n}{r^n} : n \geqslant 0 \right\}.$$

Let $\lambda > 1$, and 2λ be integer. The discrete series of G_r is the following irreducible projective unitary representation of G_r (cf [7, 8]):

$$[T_{r,\lambda}(g)f](z) = (-cz+a)^{-\lambda} f(g^{-1}z) \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_r \quad f \in H^2_\lambda(D_r).$$

The Lie algebra of G_r is generated by

$$X_- = A\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A^{-1} \qquad X_0 = A\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} A^{-1} \qquad X_+ = A\begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} A^{-1}$$

which are essentially Pauli matrices. They are exponentiated to one parameter subgroups of G_r as follows:

$$\mathrm{e}^{tX_-} = \begin{pmatrix} \cosh t & r\sinh t \\ \sinh t/r & \cosh t \end{pmatrix}$$

$$\mathrm{e}^{tX_0} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}t} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}t} \end{pmatrix}$$

$$\mathrm{e}^{tX_+} = \begin{pmatrix} \cosh t & \mathrm{i}r\sinh t \\ -\mathrm{i}\sinh t/r & \cosh t \end{pmatrix}.$$

The derived representation of the Lie algebra of G_r induced by the discrete series is

$$\mathrm{d}T_{r,\lambda}(X_-) = \frac{\mathrm{d}}{\mathrm{d}t} T_{r,\lambda}(\mathrm{e}^{tX_-})|_{t=0} = \frac{\lambda}{r} z + \frac{1}{r} z^2 \frac{\partial}{\partial z} - r\frac{\partial}{\partial z}$$

$$\mathrm{d}T_{r,\lambda}(X_0) = \frac{\mathrm{d}}{\mathrm{d}t} T_{r,\lambda}(\mathrm{e}^{tX_0})|_{t=0} = -\mathrm{i}\left(\lambda + 2z\frac{\partial}{\partial z}\right)$$

$$\mathrm{d}T_{r,\lambda}(X_+) = \frac{\mathrm{d}}{\mathrm{d}t} T_{r,\lambda}(\mathrm{e}^{tX_+})|_{t=0} = -\mathrm{i}\left(\frac{\lambda}{r} z + \frac{1}{r} z^2 \frac{\partial}{\partial z} + r\frac{\partial}{\partial z}\right).$$

$Q = \mathrm{i}\,\mathrm{d}T_{r,\lambda}(X_+)$ is understood as the position observable and $P = \mathrm{i}\,\mathrm{d}T_{r,\lambda}(X_-)$ as the conjugate momentum observable. Let

$$A^* = \frac{1}{2}(Q - \mathrm{i}P) = \frac{\lambda}{r} z + \frac{1}{r} z^2 \frac{\partial}{\partial z} \qquad A = \frac{1}{2}(Q + \mathrm{i}P) = r\frac{\partial}{\partial z}$$

then A^* and A are mutually adjoint on $H^2_\lambda(D_r)$ and can be interpreted as creation and annihilation operators respectively. Set

$$A^*_{r,\lambda} \mathrel{\widehat{=}} \frac{1}{r} A^* = \frac{\lambda}{r^2} z + \frac{1}{r^2} z^2 \frac{\partial}{\partial z} \qquad A_{r,\lambda} \mathrel{\widehat{=}} \frac{1}{r} A = \frac{\partial}{\partial z} \qquad N \mathrel{\widehat{=}} z\frac{\partial}{\partial z}$$

then

$$[A_{r,\lambda},\ A^*_{r,\lambda}] = \frac{\lambda}{r^2} + \frac{2}{r^2} N \qquad [A_{r,\lambda},\ N] = A_{r,\lambda} \qquad [A^*_{r,\lambda},\ N] = -A^*_{r,\lambda}.$$

In analogy with $\{H^2(C),\ \partial,\ \partial^*,\ H\}$, we may call $\{H^2_\lambda(D_r), A_{r,\lambda}, A^*_{r,\lambda}, N\}$ the harmonic oscillator on D_r. The latter may be viewed as a relativistic analogue of the former, the parameter r playing the role of speed of light. Note that while $\{A_{r,\lambda}, A^*_{r,\lambda}\}$ does not satisfy the CCR, $\{A_{r,\lambda}, A^*_{r,\lambda}, N\}$ satisfies the Wigner commutation relations.

Let

$$H_{r,\lambda} = A^*_{r,\lambda} A_{r,\lambda} = \frac{\lambda}{r^2} z \frac{\partial}{\partial z} + \frac{1}{r^2} z^2 \frac{\partial^2}{\partial z^2}$$

then

$$H_{r,\lambda} z^n = \frac{1}{r^2} n(\lambda + n - 1) z^n \qquad n = 0, 1, \dots\ .$$

$H_{r,\lambda}$ may be interpreted as a Hamiltonian. We shall show that $H_{r,\lambda}$ is closely related with the ultraspherical operator

$$L_\alpha = -(1 - x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (2\alpha + 1)x\frac{\mathrm{d}}{\mathrm{d}x}$$

on $L^2((-1,1), \nu_\alpha)$, where $\nu_\alpha(\mathrm{d}x) = \left(\Gamma(\alpha+1)/\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})\right)(1-x^2)^{\alpha-1/2}\mathrm{d}x$ is a probability measure on $(-1,1)$. For $\alpha = 1$, $\nu_1(\mathrm{d}x) = (2/\pi)(1-x^2)^{1/2}\mathrm{d}x$ is the famous Wigner semicircle law occurring in random matrices.

Let $\{Y_n^{(\alpha)}\}$ be the ultraspherical (Gegenbauer) polynomials defined by the generating function

$$(1 - 2xz + z^2)^{-\alpha} = \sum_{n=0}^{\infty} Y_n^{(\alpha)}(x) z^n \qquad |z| < 1.$$

Then L_α is diagonalized by $\{Y_n^{(\alpha)}\} : L_\alpha Y_n^{(\alpha)} = n(n+2\alpha)Y_n^{(\alpha)}$. Consequently $H_{1,\lambda}$ and $L_{(\lambda-1)/2}$ have the same spectrum. In the remainder of this section, we take $r = 1$, $\alpha = (\lambda - 1)/2$.

Note that

$$\left\{ h_k^{(\alpha)}(x) \,\widehat{=}\, \sqrt{\frac{\Gamma(2\alpha)k!(k+\alpha)}{\alpha\Gamma(k+2\alpha)}}\, Y_k^{(\alpha)}(x) : k = 0, 1, \dots \right\}$$

constitutes an orthonormal basis for $L^2((-1,1), \nu_\lambda)$ (cf [5] or [6]). Define the integral transform

$$Bf(z) = \int_{(-1,1)} p_\alpha(z, x) f(x) \nu_\alpha(\mathrm{d}x) \qquad f \in L^2((-1,1), \nu_\lambda) \quad z \in D_1$$

where

$$p_\alpha(z, x) = \frac{1}{\sqrt{2}} \frac{1 - z^2}{(1 - 2zx + z^2)^{\alpha+1}}$$

is the Poisson kernel function (modulo the factor $1/\sqrt{2}$) (see [6]).

Theorem 1. B is a bijection from $L^2((-1,1), \nu_\alpha)$ onto $H^2_\lambda(D_1)$ (note $\alpha = (\lambda - 1)/2$). Moreover

$$\tfrac{1}{2}\|f\| \leqslant \|Bf\| \leqslant \|f\| \qquad \forall\ f \in L^2((-1,1), \nu_\alpha)$$

and $Bh_k^{(\alpha)} = \sqrt{(k+\alpha)/(k+2\alpha)}\, e_k$, $k = 0, 1, \dots$. $H_{1,\lambda}$ and L_α are interwined by B, i.e. $B^{-1} H_{1,\lambda} B = L_\alpha$. Here $\|\cdot\|$ denotes the norm on $L^2((-1,1), \nu_\alpha)$ and $H^2_\lambda(D_1)$.

Proof. Utilizing $z\,\partial/\partial z$ acting on both sides of

$$\frac{1}{(1-2zx+z^2)^{\alpha}} = \sum_k Y_k^{(\alpha)}(x) z^k$$

leads to

$$\frac{\alpha(2zx-2z^2)}{(1-2zx+z^2)^{\alpha+1}} = \sum_k k Y_k^{(\alpha)}(x) z^k.$$

Adding $\alpha/(1-2zx+z^2)^{\alpha}$ to both sides, we obtain

$$\alpha \frac{1-z^2}{(1-2zx+z^2)^{\alpha+1}} = \sum_k (k+\alpha) Y_k^{(\alpha)}(x) z^k.$$

Hence

$$\begin{aligned} p_\alpha(z,x) &= \frac{1}{\sqrt{2}\alpha} \sum_k (k+\alpha) Y_k^{(\alpha)}(x) z^k \\ &= \frac{1}{\sqrt{2}\alpha} \sum_k \sqrt{\frac{k+\alpha}{k+2\alpha}} \sqrt{(k+\alpha)(k+2\alpha)}\, Y_k^{(\alpha)}(x) z^k \\ &= \sum_k \sqrt{\frac{k+\alpha}{k+2\alpha}}\, h_k^{(\alpha)}(x) e_k(z). \end{aligned}$$

Noting that $\{h_k^{(\alpha)}\}$ and $\{e_k\}$ constitute orthonormal bases of $L^2((-1,1),\nu_\alpha)$ and $H_\lambda^2(D_1)$, respectively, the conclusion easily follows. □

Remark. B is reminiscent of the coherent state transform introduced by Bargmann [1] in interwining the Schrödinger and Fock models of the harmonic oscillator. Though B is not unitary, its kernel is the Poisson kernel, which is widely used in complex analysis, in particular in solving Dirichilet problems [6]; hence is quite natural from a mathematical point of view.

3. Hypercontractivity

For $t > 0$, let $T_t = \mathrm{e}^{-tN}$, $S_t = \mathrm{e}^{-tH_{r,\lambda}}$ be the semigroup generated by N and $H_{r,\lambda}$ respectively. Inspired by the Euclidean case [11], we may expect that T_t is a contraction from $H_\lambda^p(D_r)$ to $H_\lambda^q(D_r)$ whenever $\mathrm{e}^{-2t} \leqslant p/q$. Unfortunately, we have not found a proof at present. However, the particular case $p = 2$, $q = 4$ (which is usually enough for applications [10]) can easily be proved as follows.

Theorem 2. If $\mathrm{e}^{-2t} \leqslant 2/4$, then $T_t : H_\lambda^2(D_r) \to H_\lambda^4(D_r)$ is a contraction, and the maximizer can only be attained at constant functions. Similarly, If $\mathrm{e}^{-2(\lambda/r^2)t} \leqslant 2/4$, then $S_t : H_\lambda^2(D_r) \to H_\lambda^4(D_r)$ is a contraction, and the maximizer can only be attained at constant functions.

Proof. Clearly, $T_t f(z) = f(\mathrm{e}^{-t} z)$. Note that

$$\left\{ e_n(z) = \sqrt{\frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)}} \frac{z^n}{r^n} : n \geqslant 0 \right\}$$

is an orthonormal basis of $H^2_\lambda(D_r)$, if $f \in H^2_\lambda(D_r)$ has the Fourier decomposition

$$f(z) = \sum_{n=0}^{\infty} a_n e_n(z)$$

then

$$\begin{aligned}
\|T_t f\|_4^4 &= \int_{D_r} (T_t f(z)\overline{T_t f(z)})^2 \,\mathrm{d}\mu_{r,\lambda}(z,\bar{z}) \\
&= \int_{D_r} \left(\sum_{m,n=0}^{\infty} \mathrm{e}^{-(m+n)t} a_m \bar{a}_n e_m(z)\overline{e_n(z)}\right)^2 \mathrm{d}\mu_{r,\lambda}(z,\bar{z}) \\
&= \int_{D_r} \sum_{m,n,j,k=0}^{\infty} \mathrm{e}^{-(m+n+j+k)t} a_m a_j \overline{a_n a_k} e_m(z) e_j(z) \overline{e_n(z) e_k(z)} \,\mathrm{d}\mu_{r,\lambda}(z,\bar{z}) \\
&= \sum_{l=0}^{\infty} \mathrm{e}^{-2lt} \sum_{\substack{m+j=l\\ n+k=l}} a_m a_j \overline{a_n a_k} \left(\frac{\Gamma(m+\lambda)}{m!\Gamma(\lambda)} \frac{\Gamma(j+\lambda)}{j!\Gamma(\lambda)} \frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)} \frac{\Gamma(k+\lambda)}{k!\Gamma(\lambda)}\right)^{1/2} \\
&\quad \times \frac{l!\Gamma(\lambda)}{\Gamma(l+\lambda)} \\
&= \sum_{l=0}^{\infty} \mathrm{e}^{-2tl} \left| \sum_{m+j=l} a_m a_j \left(\frac{l!\Gamma(\lambda)}{\Gamma(l+\lambda)} \frac{\Gamma(m+\lambda)}{m!\Gamma(\lambda)} \frac{\Gamma(j+\lambda)}{j!\Gamma(\lambda)}\right)^{1/2} \right|^2 \\
&\leqslant \sum_{l=0}^{\infty} 2^{-l} \left(\sum_{m+j=l} \frac{l!}{m!j!} \frac{\Gamma(m+\lambda)\Gamma(j+\lambda)}{\Gamma(\lambda)\Gamma(l+\lambda)}\right) \left(\sum_{m+j=l} |a_m a_j|^2\right) \\
&\leqslant \sum_{l=0}^{\infty} 2^{-l} \left(\sum_{m+j=l} \frac{l!}{m!j!}\right) \left(\sum_{m+j=l} |a_m a_j|^2\right) \\
&= \|f\|_2^4.
\end{aligned}$$

From the calculation, it is clear that $\|T_t f\|_4 = \|f\|_2$ iff f is a constant.

The second statement is proved similarly. □

4. The large-radius and large-spin limit

The following correspondences relate the harmonic oscillator on a Euclidean space with that on a Poincaré disc.

Let $r \to \infty$, $\lambda \to \infty$ and $\lambda/r^2 \to 1$, then

(1) $D_r \longrightarrow C$,
(2) $G_r \longrightarrow E(C)$ (Euclidean motion group on C),
(3) $\mathrm{d}\mu_r \longrightarrow \mathrm{d}z\mathrm{d}\bar{z}$ (Lebesgue measure on C),
(4) $\mathrm{d}\mu_{r,\lambda} \longrightarrow \pi^{-1}\mathrm{e}^{-z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}$ (Gaussian measure on C),
(5) $H^2_\lambda(D_r) \longrightarrow H^2(C)$,
(6) $A_{r,\lambda} \longrightarrow \partial$, $A^*_{r,\lambda} \longrightarrow \partial^*$, $H_{r,\lambda} \longrightarrow H = N$.

5. Discussion

In the construction of a harmonic oscillator on the Poincaré disc, two Hamiltonians naturally arise: one is $N = z\,\partial/\partial z$, which is formally the same as the free Hamiltonian of the ordinary harmonic oscillator (though they act on different Hilbert spaces); the other is $H_{r,\lambda}$ which is closely related to the ultraspherical operator widely used in special function theory and mathematical physics.

The Poincaré disc is the simplest non-Euclidean space of hyperbolic nature, the radius r may be formally interpreted as a coupling constant (e.g. the speed of light). The parameter λ occurring in the discrete series of $\mathrm{SU}(1,1)$ is the spin. Thus, the model we constructed is a two-parameter deformation of the ordinary harmonic oscillator and may serve as a simple approximation for the description of certain interacting or relativistic quantum systems. The limit $\lambda/r^2 \longrightarrow 1$ is like a non-relativistic limit.

The method can also be applied to other series representations of G_r, e.g., the principal series and complementary series [2, 7], and has a direct generalization to higher dimensions, with the Poincaré disc being replaced by the Hermitian symmetric space.

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